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Note on a paper by A. Granville and K. Soundararajan

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Abstract

In this note, we improve some results of Granville and Soundararajan on the distribution of values of the truncated random Euler product $L(1, X; y) := \prod_{p \leqslant y}(1 - X(p)/p)^{-1}$, where the $X(p)$ are independent random variables, taking the values ± 1 with equal probability $p/2(p+1)$ and 0 with probability $1/(p+1)$.

MSC: 11M20

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1. Introduction

Let d be a fundamental discriminant and χ_d the primitive real character associated to the modulus $|d|$. The study on the distribution of the values $L(1, \chi_d)$ originated with the work of Littlewood [7] and has been extended by many authors such as Chowla, Erdős, Bateman, Barban, Elliott, Joshi, Shanks, Montgomery, Vaughan, Granville, Soundararajan, etc. The reader can find a detailed historical description in [4,8]. In particular Littlewood [7] proved that under the Generalized Riemann Hypothesis one has

$$\{1 + o(1)\}/(12\pi^{-2}e^{\gamma} \log_2 |d|) \leqslant L(1, \chi_d) \leqslant \{1 + o(1)\}2e^{\gamma} \log_2 |d|, \tag{1.1}$$

where $\log_k$ denotes the k-fold iterated logarithm and γ is the Euler constant. In the opposite direction, Chowla [1] shew that there are χ_{d_1} and χ_{d_2} such that

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$$L(1,\chi_{d_1}) \geqslant \{1+o(1)\}e^\gamma \log_2|d_1|, \tag{1.2}$$

$$L(1,\chi_{d_2}) \leqslant \{1+o(1)\}\big/\big(6\pi^{-2}e^\gamma \log_2|d_2|\big). \tag{1.3}$$

Only the factor 2 in (1.1) remains in doubt on either side. Very recently Montgomery and Vaughan [8] returned to this problem and initiated a finer study of these extreme values. Write

$$\log L(1,\chi_d) = \sum_p \frac{\chi_d(p)}{p} + \sum_p \sum_{\nu\geqslant 2} \frac{\chi_d(p)^\nu}{\nu p^\nu}.$$

Since the second sum on the right-hand side is bounded, it is sufficient to consider how large, in positive and negative directions, the first sum attains. For a typical d, one may expect that it behaves like

$$\sum_p X_p/p,$$

where the X_p are independent random variables taking ± 1 with equal probability. Extrapolating this model, they proposed three conjectures on the distribution of the values of $L(1,\chi_d)$. The following is the first one.

Conjecture *(Montgomery–Vaughan). The proportion of fundamental discriminants $|d| \leqslant x$ with $L(1,\chi_d) \geqslant e^\gamma \log_2|d|$ is $> \exp(-C\log x/\log_2 x)$ and $< \exp(-c\log x/\log_2 x)$ for appropriate constants $0<c<C<\infty$. Similar estimates apply to the proportion of fundamental discriminants $|d| \leqslant x$ with $L(1,\chi_d) \leqslant 1/(6\pi^{-2}e^\gamma\log_2|d|)$.*

This conjecture has been proved very recently by Granville and Soundararajan [4]. To this end, they introduced a new probability model: For each prime p, let $X(p) = X(p,\omega)$ denote independent random variables on the probability space (Ω,μ), taking the values ± 1 with equal probability $p/2(p+1)$ and 0 with probability $1/(p+1)$. Define the random Euler product

$$L(1,X) := \prod_p \big(1 - X(p)/p\big)^{-1}. \tag{1.4}$$

The infinite product converges with probability 1, since $\mathbb{E}\big(X(p)/p\big) = 0$ for all primes p and

$$\sum_p \mathbb{E}\big(\big(X(p)/p\big)^2\big) = \sum_p 1/p(p+1) < \infty,$$

where $\mathbb{E}(Y)$ is the expectation of the random variable Y on (Ω,μ). Further let us introduce the distributions of $L(1,X)$ and of $L(1,\chi_d)$:

$$\Phi(t) := \mathrm{Prob}\big(L(1,X) \geqslant e^\gamma t\big), \tag{1.5}$$

$$\Phi_x(t) := \Bigg(\sideset{}{^\flat}\sum_{\substack{|d|\leqslant x\\ L(1,\chi_d)>e^\gamma t}} 1\Bigg)\Bigg/\Bigg(\sideset{}{^\flat}\sum_{|d|\leqslant x} 1\Bigg), \tag{1.6}$$

where $\sum^\flat$ indicates that the sum is over fundamental discriminants. First by using the saddle-point method they proved (see [4, Proposition])

$$\Phi(t) = \exp\left\{-\frac{e^{t-\gamma_0}}{t}\left[1 + O\left(\frac{1}{t}\right)\right]\right\}, \tag{1.7}$$

where

$$\gamma_0 := \int_0^1 \frac{\tanh(t)}{t}\,\mathrm{d}t + \int_1^\infty \frac{\tanh(t) - 1}{t}\,\mathrm{d}t = 0.8187\ldots. \tag{1.8}$$

Then they compared $\Phi_x(t)$ with $\Phi(t)$ and finally shew [4, Theorem 4]

$$\Phi_x(t) = \exp\left\{-\frac{e^{t-\gamma_0}}{t}\left[1 + O\left(\frac{1}{A} + \frac{1}{t}\right)\right]\right\} \tag{1.9}$$

uniformly for $\log_2 x \geqslant A \geqslant e$ and $t \leqslant \log_2 x + \log_4 x - 20$. This implies a stronger version of Montgomery–Vaughan's conjecture on the large value of $L(1, \chi_d)$.

In order to prove (1.7), they considered a more general problem, i.e., to evaluate

$$\Phi(t, y) := \mathrm{Prob}\big(L(1, X; y) \geqslant e^\gamma t\big), \tag{1.10}$$

where $L(1, X; y)$ is the truncated random Euler product given by

$$L(1, X; y) := \prod_{p \leqslant y} \big(1 - X(p)/p\big)^{-1}.$$

Define

$$E(s, y) := \mathbb{E}\big(L(1, X; y)^s\big) \quad \text{and} \quad E_p(s) := \mathbb{E}\big(\big(1 - X(p)/p\big)^{-s}\big).$$

By the independence of $X(p)$'s and its definition, we have

$$E(s, y) = \prod_{p \leqslant y} E_p(s) \tag{1.11}$$

and

$$E_p(s) = \frac{p}{2(p+1)}\left\{\left(1 - \frac{1}{p}\right)^{-s} + \left(1 + \frac{1}{p}\right)^{-s} + \frac{2}{p}\right\}. \tag{1.12}$$

Let $\kappa_0 = \kappa_0(t, y)$ be the unique positive solution of the equation

$$\sum_{p \leqslant y} \log(1 - 1/p)^{-1} \tanh(\sigma/p) = \log t + \gamma$$

and define

$$R(\kappa, y) := \sum_{p \leqslant y} p^{-2} \cosh^{-2}(\kappa/p).$$

They proved the following estimate [4, Theorem 3.1]

$$\Phi(t, y) = \frac{E(\kappa_0, y)}{\kappa_0 \sqrt{2\pi R(\kappa_0, y)}(e^\gamma t)^{\kappa_0}} \left\{1 + O\left(\frac{t^2}{e^{t/4}}\right)\right\} \tag{1.13}$$

uniformly for $t \geqslant 3$ and $\log y \geqslant t + 1$. Further if $0 \leqslant \lambda \leqslant e^{-t}$, then

$$\Phi\big(te^{-\lambda}, y\big) - \Phi(t, y) \ll \Phi(t, y)\big(e^t \lambda + e^{3t/4} y^{-1} \log y\big). \tag{1.14}$$

The aim of this note is to improve their estimates (1.7), (1.13) and (1.14). Define

$$\phi(s, y) := \log E(s, y), \qquad \phi_n(s, y) := \frac{\partial^n \phi}{\partial s^n}(s, y) \quad (n \geqslant 0). \tag{1.15}$$

Let $\kappa = \kappa(t, y)$ be the unique positive solution of the equation

$$\phi_1(\kappa, y) = \log t + \gamma. \tag{1.16}$$

According to Lemmas 2.1 and 2.2 below, the saddle point $\kappa(t, y)$ exists when $t \geqslant 1$ and $y \geqslant 2e^t$ and we have $\kappa(t, y) \asymp e^t$ in this domain. Finally define $\sigma_n := \phi_n(\kappa, y)$ for $n \geqslant 0$. We preserve these notation for the duration of this paper.

Our results are as follows:

Theorem 1. *We have*

$$\Phi(t, y) = \frac{E(\kappa, y)}{\kappa \sqrt{2\pi \sigma_2}(e^\gamma t)^{\kappa}} \left\{1 + O\left(\frac{t}{e^t}\right)\right\} \tag{1.17}$$

uniformly for $t \geqslant 1$ and $y \geqslant 2e^t$. Further for any $\varepsilon \in (0, 1)$, we have

$$\Phi\big(te^{-\lambda}, y\big) - \Phi(t, y) \ll_\varepsilon \Phi(t, y)\big(e^t \lambda + y^{-1/\varepsilon}\big) \tag{1.18}$$

uniformly for $t \geqslant 1$, $y \geqslant 2e^t$ and $0 \leqslant \lambda \leqslant e^{-t}$.

Theorem 2. *For each integer $N \geqslant 1$, we have*

$$\Phi(t, y) = \exp\left\{-\kappa\left[\sum_{n=1}^{N} \frac{a_n}{(\log \kappa)^n} + O_N\big(R_N(\kappa, y)\big)\right]\right\}$$

uniformly for $t \geqslant 1$ and $y \geqslant 2e^t$, where[1]

[1] By convention, we define $f^{(n)}(1) = f^{(n)}(1-)$ for all integers $n \geqslant 1$.

$$a_n := \int_0^\infty \left(\frac{f(u)}{u}\right)' (\log u)^{n-1}\,\mathrm{d}u \tag{1.19}$$

with

$$f(u) := \begin{cases} \log\cosh(u) & \text{if } 0 \leqslant u \leqslant 1, \\ \log\cosh(u) - u & \text{if } u > 1. \end{cases} \tag{1.20}$$

The error term $R_N(\kappa, y)$ is given by

$$R_N(\kappa, y) := \frac{1}{(\log\kappa)^{N+1}} + \frac{\kappa}{y\log y}. \tag{1.21}$$

Corollary 3. *For each integer $N \geqslant 1$, there are computable constants $a_1^*, \dots, a_N^*$ such that the asymptotic formula*

$$\Phi(t, y) = \exp\left\{-e^{t-\gamma_0}\left[\sum_{n=1}^N \frac{a_n^*}{t^n} + O_N\big(R_N\big(e^t, y\big)\big)\right]\right\}$$

holds uniformly for $t \geqslant 1$ and $y \geqslant 2e^t$. Further we have

$$a_1^* = 1, \qquad a_2^* = \gamma_0 - \frac{\gamma_0^2}{2} - \int_0^\infty \frac{f(u)}{u^2}(\log u)\,\mathrm{d}u = 1.62\dots.$$

In particular for each integer $N \geqslant 1$, we have

$$\Phi(t) = \exp\left\{-e^{t-\gamma_0}\left[\sum_{n=1}^N \frac{a_n^*}{t^n} + O_N\left(\frac{1}{t^{N+1}}\right)\right]\right\}$$

uniformly for $t \geqslant 1$.

Remark. (i) Granville and Soundararajan also investigated Montgomery and Vaughan's conjecture for the small value of $L(1, \chi_d)$ and obtained similar estimates for

$$\Psi(t, y) := \mathrm{Prob}\big(L(1, X; y) \leqslant 1/\big(6\pi^{-2}e^\gamma t\big)\big).$$

Clearly we can also improves their corresponding estimates.

(ii) As in [4], we shall apply the saddle-point method to prove our theorems.[2] But our choice for the saddle-point is different from theirs. It will be seen that our choice is more natural and is one of key reasons for the improvement on (1.13) and (1.14). Another new idea is to use the exponential sum method to improve Lemma 3.2 of [4] (see Lemma 2.4 below). Indeed, if

[2] The saddle-point method was firstly applied to the number theory by Hildebrand and Tenenbaum [5]. Interested readers are referred to [9] for an excellent paradigm.

we further apply the Vinogradov method as in [6] instead of the simple van der Corput method (see [3]) used here, the terms $y^{\mp 1/\varepsilon}$ in (1.18) and Lemma 2.4 will be sharpened to $e^{\mp c(\log y)^2}$ with some absolute constant $c > 0$, respectively.

2. Preliminary lemmas

This section is devoted to establish some technique lemmas in the saddle-point method for our purpose.

Lemma 2.1. *For any integer $N \geqslant 1$, we have*

$$\phi_1(\sigma, y) = \log_2 \sigma + \gamma + \sum_{n=1}^{N} \frac{b_n}{(\log \sigma)^n} + O_N\big(R_N(\sigma, y)\big) \tag{2.1}$$

uniformly for $y \geqslant \sigma \geqslant 2$, where γ is the Euler constant and $R_N(\sigma, y)$ is defined as in (1.21). *The constant b_n is given by*

$$b_n := \int_0^\infty \frac{f'(u)}{u} (\log u)^{n-1}\, \mathrm{d}u. \tag{2.2}$$

In particular $b_1 = \gamma_0$.

Proof. First we prove

$$\frac{E_p'(\sigma)}{E_p(\sigma)} = \begin{cases} -\log\big(1 - \frac{1}{p}\big) + O\big(\frac{e^{-\sigma/p}}{p}\big) & \text{if } p \leqslant \sigma, \\ -\tanh\big(\frac{\sigma}{p}\big)\log\big(1 - \frac{1}{p}\big) + O\big(\frac{1}{p^2} + \frac{\sigma}{p^3}\big) & \text{if } p \geqslant \sigma^{1/2}, \end{cases} \tag{2.3}$$

where the implied constants are absolutes.

By using (1.12), a simple calculation shows that

$$\frac{E_p'(\sigma)}{E_p(\sigma)} = \frac{a^\sigma \log a + b^\sigma \log b}{a^\sigma + b^\sigma + c}, \tag{2.4}$$

where $a := (1 - 1/p)^{-1}$, $b := (1 + 1/p)^{-1}$ and $c := 2/p$.

Since $a^\sigma \geqslant e^{\sigma/p} \geqslant 1 \geqslant b^\sigma$ for any prime p and any $\sigma \geqslant 2$, we easily see, for $p \leqslant \sigma$,

$$\begin{aligned} \frac{E_p'(\sigma)}{E_p(\sigma)} &= \log a + \frac{b^\sigma \log(b/a) - c \log a}{a^\sigma + b^\sigma + c} \\ &= \log a + O\left(\frac{e^{-\sigma/p}}{p}\right). \end{aligned}$$

This proves the first estimate of (2.3).

In order to verify the second, we write, in view of (2.4),

$$\begin{aligned}\frac{E_p'(\sigma)}{E_p(\sigma)} &= \frac{a^\sigma \log a + b^\sigma \log b}{a^\sigma + b^\sigma} + O\left(\frac{e^{-\sigma/p}}{p^2}\right)\\ &= \frac{a^\sigma - b^\sigma}{a^\sigma + b^\sigma}\log a + O\left(\frac{e^{-\sigma/p}}{p^2}\right).\end{aligned}$$

If $p \geqslant \sigma^{1/2}$, we have

$$\begin{aligned}\frac{a^\sigma - b^\sigma}{a^\sigma + b^\sigma} &= \frac{e^{\sigma/p+O(\sigma/p^2)} - e^{-\sigma/p+O(\sigma/p^2)}}{e^{\sigma/p+O(\sigma/p^2)} + e^{-\sigma/p+O(\sigma/p^2)}}\\ &= \tanh\left(\frac{\sigma}{p}\right) + O\left(\frac{\sigma}{p^2}\right).\end{aligned}$$

Putting all these estimates together, we obtain the second formula of (2.3).

Now we are ready to prove (2.1). According to (1.11), we have

$$\phi_1(\sigma, y) = \sum_{p\leqslant y} E_p'(\sigma)/E_p(\sigma).$$

We use the first asymptotic formula of (2.3) for $p \leqslant \sigma^{2/3}$ and the second for $\sigma^{2/3} < p \leqslant y$ to write

$$\begin{aligned}\phi_1(\sigma, y) &= \sum_{p\leqslant\sigma^{2/3}} \log(1-1/p)^{-1} + \sum_{\sigma^{2/3}<p\leqslant y} \tanh(\sigma/p)\log(1-1/p)^{-1} + O\big(\sigma^{-1/3}\big)\\ &= \sum_{p\leqslant\sigma} \log(1-1/p)^{-1} + \sum_{\sigma^{2/3}<p\leqslant y} f'(\sigma/p)\log(1-1/p)^{-1} + O\big(\sigma^{-1/3}\big).\end{aligned}\tag{2.5}$$

With the help of the prime number theorem of form

$$\sum_{p\leqslant\sigma} \log(1-1/p)^{-1} = \log_2\sigma + \gamma + O\big(e^{-2\sqrt{\log\sigma}}\big),\tag{2.6}$$

we can write

$$\sum_{\sigma^{2/3}<p\leqslant y} f'(\sigma/p)\log(1-1/p)^{-1} = \int_{\sigma^{2/3}}^{y} \frac{f'(\sigma/t)}{t\log t}\,\mathrm{d}t + O(R_1),\tag{2.7}$$

where

$$R_1 := f'\left(\frac{\sigma}{y}\right)e^{-2\sqrt{\log y}} + f'\big(\sigma^{1/3}\big)e^{-\sqrt{\log\sigma}} + \sigma\int_{\sigma^{2/3}}^{y} \frac{|f''(\sigma/t)|}{t^2}e^{-2\sqrt{\log t}}\,\mathrm{d}t.$$

In view of the following simple facts that

$$f(u) \asymp \begin{cases} u^2 & \text{if } 0 \leqslant u \leqslant 1, \\ 1 & \text{if } u > 1, \end{cases}$$
$$f'(u) \asymp \begin{cases} u & \text{if } 0 \leqslant u \leqslant 1, \\ e^{-2u} & \text{if } u > 1, \end{cases}$$
$$f''(u) \asymp \begin{cases} 1 & \text{if } 0 \leqslant u \leqslant 1, \\ e^{-2u} & \text{if } u > 1, \end{cases} \tag{2.8}$$

it is easy to deduce that

$$\begin{aligned} R &\ll \frac{\sigma}{y} e^{-\sqrt{\log y}} + e^{-2\sigma^{1/3}} + \sigma \int_{\sigma^{2/3}}^{\sigma} e^{-2\sigma/t - 2\sqrt{\log t}} \frac{\mathrm{d}t}{t^2} + \sigma \int_{\sigma}^{y} e^{-2\sqrt{\log t}} \frac{\mathrm{d}t}{t^2} \\ &\ll \frac{\sigma}{y} e^{-\sqrt{\log y}} + e^{-\sqrt{\log \sigma}}. \end{aligned}$$

In order to evaluate the integral of (2.7), we use the change of variable $u = \sigma/t$ to write

$$\begin{aligned} \int_{\sigma^{2/3}}^{y} \frac{f'(\sigma/t)}{t \log t} \mathrm{d}t &= \int_{\sigma/y}^{\sigma^{1/3}} \frac{f'(u)}{u \log(\sigma/u)} \mathrm{d}u \\ &= \int_{\sigma^{-1/3}}^{\sigma^{1/3}} \frac{f'(u)}{u \log(\sigma/u)} \mathrm{d}u + O\big(R_1'\big), \end{aligned}$$

where

$$\begin{aligned} R_1' &:= \int_0^{\sigma^{-1/3}} \frac{|f'(u)|}{u \log(\sigma/u)} \mathrm{d}u + \int_0^{\sigma/y} \frac{|f'(u)|}{u \log(\sigma/u)} \mathrm{d}u \\ &\ll \int_0^{\sigma^{-1/3}} \frac{\mathrm{d}u}{\log(\sigma/u)} + \int_0^{\sigma/y} \frac{\mathrm{d}u}{\log(\sigma/u)} \\ &\ll \frac{1}{\sigma^{1/3} \log \sigma} + \frac{\sigma}{y \log y}. \end{aligned}$$

On the other hand, we have

$$\int_{\sigma^{-1/3}}^{\sigma^{1/3}} \frac{f'(u)}{u \log(\sigma/u)} \mathrm{d}u = \sum_{n=1}^{N} \frac{b_n(\sigma)}{(\log \sigma)^n} + O_N\left(\frac{b_{N+1}(\sigma)}{(\log \sigma)^{N+1}}\right),$$

where

$$\begin{aligned} b_n(\sigma) &:= \int_{\sigma^{-1/3}}^{\sigma^{1/3}} \frac{f'(u)}{u} (\log u)^{n-1}\,\mathrm{d}u \\ &= b_n + O\biggl(\frac{(\log\sigma)^{n-1}}{\sigma^{1/3}}\biggr). \end{aligned}$$

Inserting these estimates into (2.7), we find that

$$\sum_{\sigma^{2/3}<p\leqslant y} g\biggl(\frac{\sigma}{p}\biggr) \log\biggl(1-\frac{1}{p}\biggr)^{-1} = \sum_{n=1}^{N} \frac{b_n}{(\log\sigma)^n} + O_N\bigl(R_N(\sigma, y)\bigr). \tag{2.9}$$

Now the required result follows from (2.5), (2.6) and (2.9). □

Lemma 2.2. *For each integer $N \geqslant 1$, there are computable constants $c_1, \ldots, c_N$ such that the asymptotic formula*

$$\kappa(t, y) = e^{t-\gamma_0} \Biggl\{1 + \sum_{n=1}^{N} \frac{c_n}{t^n} + O_N\bigl(R_N^*(t, y)\bigr)\Biggr\} \tag{2.10}$$

holds uniformly for $t \geqslant 1$ and $y \geqslant 2e^t$, where

$$R_N^*(t, y) := \frac{1}{t^{N+1}} + \frac{e^t t}{y \log y}.$$

Further we have $c_1 = -\frac{1}{2}b_1^2 - b_2$.

Proof. By Lemma 2.1 and (1.16), we have

$$\log t = \log_2 \kappa + \sum_{n=1}^{N+1} \frac{b_n}{(\log\kappa)^n} + O_N\bigl(R_{N+1}(\kappa, y)\bigr), \tag{2.11}$$

where $R_N(\kappa, y)$ is defined as in (1.21). Clearly this implies

$$\log\kappa = t + O(1) \tag{2.12}$$

and

$$\begin{aligned} t &= (\log\kappa) \prod_{n=1}^{N+1} \exp\biggl\{\frac{b_n}{(\log\kappa)^n}\biggr\} \exp\bigl\{O_N\bigl(R_{N+1}(\kappa, y)\bigr)\bigr\} \\ &= (\log\kappa) \prod_{n=1}^{N+1} \Biggl\{\sum_{m_n=0}^{N+1} \frac{1}{m_n!}\biggl(\frac{b_n}{(\log\kappa)^n}\biggr)^{m_n} + O_N\bigl(R_{N+1}(\kappa, y)\bigr)\Biggr\}. \end{aligned}$$

Developping this product, it follows that

$$t=(\log\kappa)\left\{\sum_{n=0}^{N+1}\frac{b_n'}{(\log\kappa)^n}+O_N\big(R_{N+1}(\kappa,y)\big)\right\},\tag{2.13}$$

where

$$b_n':=\sum_{\substack{m_1\geqslant 0,\dots,m_{N+1}\geqslant 0\\ m_1+2m_2+\cdots+(N+1)m_{N+1}=n}}\frac{b_1^{m_1}\cdots b_{N+1}^{m_{N+1}}}{m_1!\cdots m_{N+1}!}.$$

Since $b_0'=1$ and $b_1'=\gamma_0$, the preceding asymptotic formula can be written, in view of (2.12), as

$$t=\log\kappa+\gamma_0+\sum_{n=1}^{N}\frac{b_{n+1}'}{(\log\kappa)^n}+O_N\big(tR_{N+1}\big(e^t,y\big)\big).\tag{2.14}$$

With the help of (2.14), a simple recurrence leads to

$$t=\log\kappa+\gamma_0+\sum_{n=1}^{N}\frac{\gamma_n}{t^n}+O_N\big(R_N^*(t,y)\big),\tag{2.15}$$

where the γ_n are constants. In particular we have $\gamma_1=b_2'=\frac{1}{2}b_1^2+b_2$.

In fact taking $N=0$ in (2.14), we see that (2.15) holds for $N=0$. Suppose that it holds for $1,\dots,N-1$. Inserting these into (2.14), we find

$$t=\log\kappa+\gamma_0+\sum_{n=1}^{N}\frac{b_{n+1}'}{t^n}\left\{1-\sum_{i=1}^{N-n}\frac{\gamma_{i-1}}{t^i}+O_N\left(\frac{R_{N-n-1}^*(t,y)}{t}\right)\right\}^{-n}+O_N\big(R_N^*(t,y)\big)$$

with the convention $R_{-1}^*(t,y):=1$. Obviously the preceding estimate implies (2.15).

Now the result of Lemma 2.2 is an immediate consequence of (2.15) with

$$c_n:=\sum_{\substack{m_1\geqslant 0,\dots,m_N\geqslant 0\\ m_1+2m_2+\cdots+Nm_N=n}}(-1)^{m_1+\cdots+m_N}\frac{\gamma_1^{m_1}\cdots\gamma_N^{m_N}}{m_1!\cdots m_N!}.$$

This completes the proof. □

In the next lemma, we estimate σ_j. This is necessary for controlling the error terms in the proof of our theorems.

Lemma 2.3. *For $t\geqslant 1$ and $y\geqslant 2e^t$, we have*

$$\sigma_1=\log_2\kappa+O(1),\tag{2.16}$$

$$\sigma_2=\frac{1}{\kappa\log\kappa}+O\left(\frac{1}{\kappa(\log\kappa)^2}+\frac{1}{y\log y}\right),\tag{2.17}$$

$$\sigma_j\ll 1/\big(\kappa^{j-1}\log\kappa\big)\quad (j=3,4).\tag{2.18}$$

Proof. The first estimate follows immediately from the definition of σ_1 and (2.1).

In order to prove (2.17), we write, in view of (2.4),

$$\begin{aligned}\left(\frac{E_p'(\kappa)}{E_p(\kappa)}\right)' &= \frac{(ab)^\kappa(\log a-\log b)^2+c(a^\kappa(\log a)^2+b^\kappa(\log b)^2)}{(a^\kappa+b^\kappa+c)^2}\\ &= \frac{(ab)^\kappa(\log a-\log b)^2}{(a^\kappa+b^\kappa)^2}+O\left(\frac{e^{-\kappa/p}}{p^3}\right)\\ &= \frac{4(ab)^\kappa}{p^2(a^\kappa+b^\kappa)^2}+O\left(\frac{e^{-\kappa/p}}{p^3}\right). \end{aligned} \tag{2.19}$$

If $p \geqslant \kappa^{1/2}$, then

$$\begin{aligned}\frac{4(ab)^\kappa}{p^2(a^\kappa+b^\kappa)^2} &= \frac{4e^{O(\kappa/p^2)}}{p^2(e^{\kappa/p+O(\kappa/p^2)}+e^{-\kappa/p+O(\kappa/p^2)})^2}\\ &= \frac{4}{p^2(e^{\kappa/p}+e^{-\kappa/p})^2}\left\{1+O\left(\frac{\kappa}{p^2}\right)\right\}\\ &= \frac{1}{p^2\cosh^2(\kappa/p)}+O\left(\frac{e^{-\kappa/p}}{p^3}\right).\end{aligned}$$

On the other hand, for $p \leqslant \kappa^{1/2}$ we have

$$\frac{(ab)^\kappa}{p^2(a^\kappa+b^\kappa)^2}\leqslant\frac{1}{p^2(a/b)^\kappa}=\frac{e^{-\kappa\log(a/b)}}{p^2}\leqslant\frac{e^{-2\kappa/p}}{p^2}\ll\frac{e^{-\kappa/p}}{p^3}.$$

Similarly we can prove, for $p \leqslant \kappa^{1/2}$,

$$\frac{1}{p^2\cosh^2(\kappa/p)}\ll\frac{e^{-2\kappa/p}}{p^2}\ll\frac{e^{-\kappa/p}}{p^3}.$$

Thus we can write, in all cases,

$$\frac{4(ab)^\kappa}{p^2(a^\kappa+b^\kappa)^2}=\frac{1}{p^2\cosh^2(\kappa/p)}+O\left(\frac{e^{-\kappa/p}}{p^3}\right).$$

Inserting it into (2.19), it follows that

$$\left(\frac{E_p'(\kappa)}{E_p(\kappa)}\right)'=\frac{1}{p^2\cosh^2(\kappa/p)}+O\left(\frac{e^{-\kappa/p}}{p^3}\right).$$

From this we deduce easily, as before, asymptotic formula (2.17).

We have

$$\begin{aligned}
&\left(\frac{E_p'(\kappa)}{E_p(\kappa)}\right)'' \\
&\quad = -\frac{(ab)^\kappa \log^3(a/b)(a^\kappa - b^\kappa)}{(a^\kappa + b^\kappa + c)^3} + c^2 \frac{a^\kappa (\log a)^3 + b^\kappa (\log b)^3}{(a^\kappa + b^\kappa + c)^3} \\
&\qquad - c \frac{a^\kappa (a^\kappa - 2b^\kappa)(\log a)^3 + b^\kappa (b^\kappa - 2a^\kappa)(\log b)^3 + 3(ab)^\kappa (\log a)(\log b) \log(ab)}{(a^\kappa + b^\kappa + c)^3}.
\end{aligned}$$

Since $a^\sigma \geqslant e^{\sigma/p} \geqslant 1 \geqslant b^\sigma$, it is apparent that

$$\left(\frac{E_p'(\kappa)}{E_p(\kappa)}\right)'' \ll \frac{e^{-\kappa/p}}{p^3}.$$

From this we deduce

$$\sigma_3 \ll \int_2^y \frac{e^{-\kappa/u}}{u^3 \log u}\,\mathrm{d}u \ll \frac{1}{\kappa^3} \int_2^{\kappa/3\log\kappa} \frac{\mathrm{d}u}{u^3 \log u} + \frac{1}{\kappa^2} \int_{\kappa/y}^{3\log\kappa} \frac{v e^{-v}}{\log(\kappa/v)}\,\mathrm{d}v \ll \frac{1}{\kappa^2 \log\kappa}.$$

Similarly we can prove $\sigma_4 \ll 1/\kappa^3 \log\kappa$. This completes the proof. □

The third lemma is an improvement of [4, Lemma 3.2].

Lemma 2.4. *For any $\varepsilon \in (0, 1)$, we have*

$$\left|\frac{E(\kappa + i\tau, y)}{E(\kappa, y)}\right| \leqslant \begin{cases} e^{-c_1\tau^2/(\kappa\log\kappa)} & \text{if } |\tau| \leqslant \kappa, \\ e^{-c_1|\tau|/\log|\tau|} & \text{if } \kappa \leqslant |\tau| \leqslant y, \\ e^{-c_2(\varepsilon) y/\log y} & \text{if } y \leqslant |\tau| \leqslant y^{1/\varepsilon}, \\ 1 & \text{if } |\tau| \geqslant y^{1/\varepsilon}, \end{cases}$$

uniformly for $t \geqslant 1$ and $y \geqslant 2e^t$, where $c_1 > 0$ is an absolute constant and $c_2(\varepsilon) > 0$ a constant depending on ε only.

Proof. With the help of (1.12), we can write

$$E_p(\kappa + i\tau) = (1 - 1/p)^{-i\tau}\bigl\{r_1 + r_2 e^{i\tau\log((p-1)/(p+1))} + r_3 e^{i\tau\log(1-1/p)}\bigr\}, \tag{2.20}$$

where

$$r_1 := \frac{p}{2(p+1)}(1 - 1/p)^{-\kappa}, \qquad r_2 := \frac{p}{2(p+1)}(1 + 1/p)^{-\kappa}, \qquad r_3 := \frac{1}{p+1}.$$

A simple calculation shows that

$$\bigl|E_p(\kappa + i\tau)\bigr|^2 \leqslant (r_1 + r_2 + r_3)^2 - 2r_1 r_2\bigl[1 - \cos\bigl(\tau f(p)\bigr)\bigr],$$

where $f(x) := \log((x+1)/(x-1))$. In view of $1 - x \leqslant e^{-x}$ for $0 \leqslant x \leqslant 1$, we deduce

$$\left|E_p(\kappa + i\tau)\right| \leqslant \left|E_p(\kappa)\right| \exp\left\{-\frac{r_1 r_2}{(r_1+r_2+r_3)^2}\left[1-\cos\left(\tau f(p)\right)\right]\right\} \tag{2.21}$$

for all primes p. Clearly for $p \geqslant \kappa$, we have $r_1 r_3/(r_1+r_2+r_3)^2 \asymp 1$. Thus there is an absolute constant $c > 0$ such that

$$\left|E(s, y)\right| \leqslant \left|E(\kappa, y)\right| e^{-cS_\tau(\kappa, y)}, \tag{2.22}$$

where

$$S_\tau(\kappa, y) := \sum_{\kappa \leqslant p \leqslant y} \left(1 - \cos\left[\tau f(p)\right]\right).$$

Next we give the required lower bounds for $S_\tau(\kappa, y)$ according the size of $|\tau|$.

(i) *The case of* $|\tau| \leqslant \kappa$.

For all primes p in $(\kappa/2, \kappa)$ we have

$$1 - \cos\left[\tau f(p)\right] \geqslant 2\pi^{-2}\left[\tau f(p)\right]^2 \gg |\tau|^2/p^2,$$

which implies

$$S_\tau(\kappa, y) \geqslant c \sum_{\kappa/2 \leqslant p \leqslant \kappa} |\tau|^2/p^2 \geqslant c|\tau|^2/(\kappa \log \kappa).$$

(ii) *The case of* $\kappa \leqslant |\tau| \leqslant y$.

We apply the above argument with the primes in $(|\tau|, 2|\tau|)$ getting the desired lower bound.

(iii) *The case of* $y \leqslant |\tau| \leqslant y^{3/2}$.

We let $\delta := 10^{-10}$ and divide the interval $(y/2, y)$ into subintervals of length $\delta y^2/|\tau|$ (with possibly the last interval possibly being shorter). Call an interval good if $\cos[\tau f(p)] \leqslant \cos(\delta/10)$ for all primes p in that interval, and bad otherwise.

For every prime p in a bad interval, then there exists a unique positive integer ℓ such that $|\tau f(p) - 2\pi\ell| \leqslant \delta/10$. From this, we have

$$\ell \leqslant \left(|\tau| f(p) + \delta/10\right)/2\pi \leqslant |\tau|/y. \tag{2.23}$$

Let $p_1 > p_2$ be two primes in the bad intervals corresponding to the same integer ℓ, then

$$\left|\tau f(p_1) - \tau f(p_2)\right| \leqslant \left|\tau f(p_1) - 2\pi\ell\right| + \left|\tau f(p_2) - 2\pi\ell\right| \leqslant \delta/5.$$

On the other hand, we have

$$\begin{aligned}\left|\tau f(p_1)-\tau f(p_2)\right| &\geqslant |\tau|(p_1-p_2)/(p_1-1)(p_2+1)\\ &\geqslant |\tau|(p_1-p_2)/\left(2y^2\right).\end{aligned}$$

Thus $|p_1-p_2|\leqslant 2y^2\delta/5|\tau|<y^2\delta/|\tau|$. This shows that there are at most 2 bad intervals corresponding to the same integer. In view of (2.23), there are at most $2|\tau|/y$ bad intervals. According to Brun–Titchmarsh theorem, each bad interval contains at most $2(\delta y^2/|\tau|)/\log(\delta y^2/|\tau|)\leqslant 5\delta y^2/(|\tau|\log y)$ primes. Thus the number of primes in the bad intervals is at most

$$\left(|\tau|/y\right)5\delta y^2/\left(|\tau|\log y\right)=5\delta y/\log y$$

and there are at least $y/3\log y$ primes in the good intervals. For each good prime p, we have

$$1-\cos\left[\tau f(p)\right]\geqslant 1-\cos(\delta/10)\geqslant 2\left[\delta/(10\pi)\right]^2.$$

This gives the lower bound of Lemma 2.4 in this case.

(iv) *The case of* $y^{3/2}\leqslant|\tau|\leqslant y^{1/\varepsilon}$.

Let $e(u):=e^{2\pi iu}$ and define

$$S(y):=\sum_{y/2\leqslant p\leqslant y} e\left(\tau f(p)\right).$$

By integration by parts, we can write

$$\begin{aligned}S(y) &\ll \frac{1}{\log y}\sup_{y/2\leqslant t\leqslant y}\left|\sum_{y/2\leqslant p\leqslant t}(\log p)e\left(\tau f(p)\right)\right|\\ &\ll \frac{1}{\log y}\sup_{y/2\leqslant t\leqslant y}\left|\sum_{y/2\leqslant p\leqslant t}\Lambda(n)e\left(\tau f(n)\right)\right|+y^{1/2},\end{aligned}$$

where $\Lambda(n)$ is the von Mangoldt function. Using Vaughan's identity [2, (24.6)], we can write

$$S(y)\ll y^{\varepsilon}\sup_{y/2\leqslant t\leqslant y}\left(|S_{\mathrm{I}}|+|S_{\mathrm{II}}|\right)+y^{1/2}, \tag{2.24}$$

where

$$\begin{aligned}S_{\mathrm{I}} &:= \sum_{m\leqslant y^{1/3}} a_m \sum_{y/2\leqslant mn\leqslant t} e\left(\tau f(mn)\right),\\ S_{\mathrm{II}} &:= \sum_{y^{1/3}\leqslant m\leqslant y^{1/2}} a_m \sum_{y/2\leqslant mn\leqslant t} b_n e\left(\tau f(mn)\right),\end{aligned}$$

and $|a_m|\leqslant 1$ and $|b_n|\leqslant 1$.

By applying the exponent pairs (k,l) to the sum over n in S_{I} [3, Chapter 3], we deduce

$$\begin{aligned} S_{\mathrm{I}} &\ll \sum_{m\leqslant y^{1/3}} \left(\left(|\tau|m/y^2\right)^k (y/m)^l + \left(|\tau|m/y^2\right)^{-1}\right) \\ &\ll |\tau|^k y^{(1-2k+2l)/3} + |\tau|^{-1} y^2 \log y. \end{aligned}$$

This implies that for any $\delta > 0$,

$$S_{\mathrm{I}} \ll y^{1-\delta} \quad \left(y^{1+\delta}\log y \leqslant |\tau| \leqslant y^{2(1+k-l)/3k-\delta/k}\right). \tag{2.25}$$

It remains to estimate S_{II}. For this we define

$$S_{\mathrm{II}}(M,N) := \sum_{m\sim M} a_m \sum_{n\sim N} b_n e\big(\tau f(mn)\big),$$

where $y^{1/3} \leqslant M \leqslant y^{1/2}$, $MN \asymp y$ and $m \sim M$ means $M < m \leqslant 2M$.

By Lemma 2.5 of [3], it follows that

$$\left|S_{\mathrm{II}}(M,N)\right|^2 \leqslant (MN)^2 H^{-1} + MNH^{-1} \sum_{h\leqslant H}\sum_{n\sim N}\left|\sum_{m\sim M} e\Big(\tau\big[f\big(m(n+h)\big) - f(mn)\big]\Big)\right|,$$

for any $1 \leqslant H \leqslant N$. By applying the exponent pairs (k,l) to the sum over m, we deduce

$$\begin{aligned} \left|S_{\mathrm{II}}(M,N)\right|^2 &\leqslant (MN)^2 H^{-1} + MNH^{-1} \sum_{h\leqslant H}\sum_{n\sim N}\left(\left(|\tau|hn/y^3\right)^k M^l + \left(|\tau|hn/y^3\right)^{-1}\right) \\ &\leqslant \left(y^2 H^{-1} + |\tau|^k y^{1-2k+l} N^{1-l} H^k\right)\log y. \end{aligned}$$

Optimizing H over $[1, M]$ yields

$$\left|S_{\mathrm{II}}(M,N)\right|^2 \leqslant \left(\left(|\tau|^{3k} y^{5+l}\right)^{1/(3+3k)} + y^{5/3}\right)\log y.$$

This implies, for any $\delta > 0$,

$$S_{\mathrm{II}} \ll y^{1-\delta} \quad \left(y \leqslant |\tau| \leqslant y^{[1+6k-l-10(1+k)\delta]/3k}\right). \tag{2.26}$$

Inserting (2.25) and (2.26) into (2.24) and taking $(k,l) = A^{q-2}(\frac{1}{2},\frac{1}{2}) = (\frac{1}{2^q-2}, 1-\frac{q-1}{2^q-2})$, where A is the Weyl–van der Corput process [3, Chapter 3], we find that

$$S(y) \ll_q y^{1-10^{-q}} \quad \left(y^{1+10^{-q}} \leqslant |\tau| \leqslant y^{q/3}\right).$$

This implies the required result. $\square$

3. Proof of Theorem 1

We shall divide the proof in several steps which are embodied in the following lemmas. The first one is (3.6) and (3.7) of [4]. For the convenience of readers, we give here a detailed proof.

Lemma 3.1. *Let $t \geqslant 1$, $y \geqslant 2e^t$ and $0 < \lambda \leqslant e^{-t}$. Then we have*

$$\Phi(t, y) \leqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^\gamma t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s \leqslant \Phi\bigl(te^{-\lambda}, y\bigr), \tag{3.1}$$

$$\Phi\bigl(te^{-\lambda}, y\bigr) - \Phi(t, y) \leqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s, y)}{(e^\gamma t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\bigl(e^{\lambda s}-e^{-\lambda s}\bigr)\,\mathrm{d}s. \tag{3.2}$$

Proof. For any $c > 0$ and $\lambda > 0$, we have, by Perron's formula [10, Lemma II.2.1.1],

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s &= \frac{1}{\lambda}\int_0^\lambda \left(\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\bigl(ye^v\bigr)^s\frac{\mathrm{d}s}{s}\right)\mathrm{d}v \\ &= \begin{cases} 0 & \text{if } 0 < y < e^{-\lambda}, \\ 1 + (\log y)/\lambda \in [0, 1] & \text{if } e^{-\lambda} \leqslant y \leqslant 1, \\ 1 & \text{if } y > 1. \end{cases}\end{aligned} \tag{3.3}$$

Let $\mathbf{1}_{\{\omega\in\Omega:\ L(1,X;y)>e^\gamma t\}}(\omega)$ be the characteristic function of the set $\{\omega \in \Omega:\ L(1, X; y) > e^\gamma t\}$. Then by (3.3), we have

$$\mathbf{1}_{\{\omega\in\Omega:\ L(1,X;y)>e^\gamma t\}}(\omega) \leqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\left(\frac{L(1, X; y)}{e^\gamma t}\right)^s\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.$$

Integrating over Ω and interchanging the order of integrations yields

$$\begin{aligned}\Phi(t, y) &\leqslant \int_\Omega\left(\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\left(\frac{L(1, X; y)}{e^\gamma t}\right)^s\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s\right)\mathrm{d}\mu(\omega) \\ &= \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{E(s, y)}{(e^\gamma t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.\end{aligned}$$

This proves the first inequality of (3.1). The second can be treated by noticing that

$$\mathbf{1}_{\{\omega\in\Omega:\ L(1,X;y)>e^{\gamma-\lambda}t\}}(\omega) = \mathbf{1}_{\{\omega\in\Omega:\ L(1,X;y)>e^{\gamma}t\}}(\omega) + \mathbf{1}_{\{\omega\in\Omega:\ e^{\gamma}t\geqslant L(1,X;y)>e^{\gamma-\lambda}t\}}(\omega)$$
$$\geqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\left(\frac{L(1,X;y)}{e^{\gamma}t}\right)^{s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s.$$

From (3.1), we can deduce

$$\Phi\big(te^{-\lambda},y\big)-\Phi(t,y)\leqslant \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{E(s,y)}{(e^{\gamma-\lambda}t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s-\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{E(s,y)}{(e^{\gamma+\lambda}t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s$$
$$=\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{E(s,y)}{(e^{\gamma}t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\big(e^{\lambda s}-e^{-\lambda s}\big)\,\mathrm{d}s.$$

This completes the proof. □

Lemma 3.2. *Let $t\geqslant 1$, $y\geqslant 2e^t$ and $0<\kappa\lambda\leqslant 1$. Then we have*

$$\frac{1}{2\pi i}\int_{\kappa-i\kappa}^{\kappa+i\kappa}\frac{E(s,y)}{(e^{\gamma}t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s=\frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^{\gamma}t)^{\kappa}}\left\{1+O\left(\kappa\lambda+\frac{\log\kappa}{\kappa}\right)\right\}.$$

Proof. First we write, for $s=\kappa+i\tau$ and $|\tau|\leqslant\kappa$,

$$E(s,y)=\exp\left\{\sigma_0+i\sigma_1\tau-\frac{\sigma_2}{2}\tau^2-i\frac{\sigma_3}{6}\tau^3+O\big(\sigma_4\tau^4\big)\right\},$$
$$\frac{e^{\lambda s}-1}{\lambda s^2}=\frac{1}{\kappa}\left\{1-\frac{i}{\kappa}\tau+O\left(\kappa\lambda+\frac{\tau^2}{\kappa^2}\right)\right\}.$$

Since $\sigma_1=\log t+\gamma$, we have

$$\frac{E(s,y)}{(e^{\gamma}t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}=\frac{E(\kappa,y)}{\kappa(e^{\gamma}t)^{\kappa}}e^{-(\sigma_2/2)\tau^2}\left\{1-\frac{i}{\kappa}\tau-i\frac{\sigma_3}{6}\tau^3+O\big(R(\tau)\big)\right\}$$

with

$$R(\tau):=\kappa\lambda+\kappa^{-2}\tau^2+\sigma_4\tau^4+\sigma_3^2\tau^6.$$

Now we integrate the last expression over $|\tau|\leqslant\kappa$ to obtain

$$\frac{1}{2\pi i}\int_{\kappa-i\kappa}^{\kappa+i\kappa}\frac{E(s,y)}{(e^{\gamma}t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s=\frac{E(\kappa,y)}{2\pi\kappa(e^{\gamma}t)^{\kappa}}\int_{\kappa-i\kappa}^{\kappa+i\kappa}e^{-(\sigma_2/2)\tau^2}\big\{1+O\big(R(\tau)\big)\big\}\,\mathrm{d}\tau, \tag{3.4}$$

where we have used the fact that the integrals involving $(i/\kappa)\tau$ and $(i\sigma_3/6)\tau^3$ vanish.

On the other hand, by using Lemma 2.3 we have

$$\int_{\kappa-i\kappa}^{\kappa+i\kappa} e^{-(\sigma_2/2)\tau^2}\,\mathrm{d}\tau = \sqrt{\frac{2\pi}{\sigma_2}}\left\{1+O\left(\exp\left\{-\frac{1}{2}\kappa^2\sigma_2\right\}\right)\right\},$$

$$\int_{\kappa-i\kappa}^{\kappa+i\kappa} e^{-(\sigma_2/2)\tau^2} R(\tau)\,\mathrm{d}\tau \ll \frac{1}{\sqrt{\sigma_2}}\left(\kappa\lambda+\frac{1}{\kappa^2\sigma_2}+\frac{\sigma_3^2}{\sigma_2^3}+\frac{\sigma_4}{\sigma_2^2}\right)$$
$$\ll \frac{1}{\sqrt{\sigma_2}}\left(\kappa\lambda+\frac{\log\kappa}{\kappa}\right).$$

Inserting these into (3.4), we obtain the required result. □

Lemma 3.3. *For any $\varepsilon>0$, we have*

$$\int_{\kappa\pm i\kappa}^{\kappa\pm i\infty} \frac{E(s,y)}{(e^\gamma t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s \ll_\varepsilon \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^\kappa}\frac{e^{-c\kappa/\log\kappa}+y^{-1/\varepsilon}}{\lambda}, \tag{3.5}$$

$$\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{E(s,y)}{(e^\gamma t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\left(e^{\lambda s}-e^{-\lambda s}\right)\mathrm{d}s \ll_\varepsilon \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^\kappa}\left(\kappa\lambda+\frac{1}{y^{2/\varepsilon}\lambda}\right), \tag{3.6}$$

uniformly for $t\geqslant 1$, $y\geqslant 2e^t$ and $0<\kappa\lambda\leqslant 1$.

Proof. We split the integral in (3.5) into three parts according to

$$\kappa\leqslant|\tau|\leqslant y,\qquad y\leqslant|\tau|\leqslant y^{2/\varepsilon},\qquad |\tau|\geqslant y^{2/\varepsilon}.$$

By using Lemma 2.4 and the inequality $(e^{\lambda s}-1)/s^2\ll 1/\tau^2$, the integral in (3.5) is

$$\ll \frac{E(\kappa,y)}{(e^\gamma t)^\kappa\lambda}\left(\frac{e^{-c\kappa/\log\kappa}}{\kappa}+\frac{e^{-cy/\log y}}{y}+\frac{1}{y^{2/\varepsilon}}\right).$$

This implies (3.5) since $y\geqslant 2e^t\asymp\kappa$ and $\sigma_2\asymp 1/\kappa\log\kappa$.

Similarly we split the integral in (3.6) into four parts according to

$$|\tau|\leqslant\kappa,\qquad \kappa<|\tau|\leqslant y,\qquad y<|\tau|\leqslant y^{3/\varepsilon},\qquad |\tau|\geqslant y^{3/\varepsilon}.$$

By using Lemma 2.4 and the inequalities

$$\left(e^{\lambda s}-1\right)/\lambda s\ll\min\left\{1,1/\lambda|\tau|\right\},\qquad \left(e^{\lambda s}-e^{-\lambda s}\right)/s\ll\min\left\{\lambda,1/|\tau|\right\},$$

the integral in (3.6) is, as before,

$$\ll_\varepsilon \frac{E(\kappa, y)}{(e^\gamma t)^\kappa}\big(\lambda\sqrt{\kappa\log\kappa} + \lambda e^{-c\kappa/\log\kappa} + \lambda e^{-cy/\log y} + \lambda^{-1}y^{-3/\varepsilon}\big),$$

which implies (3.6), since the second and third terms can be absorbed by the first one. $\square$

Now we are ready to complete the proof of Theorem 1. From Lemmas 3.2 and 3.3, we have

$$\frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty}\frac{E(s,y)}{(e^\gamma t)^s}\frac{e^{\lambda s}-1}{\lambda s^2}\,\mathrm{d}s = \frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^\kappa}\big\{1+O(R)\big\}, \tag{3.7}$$

where

$$R := \frac{\log\kappa}{\kappa} + \kappa\lambda + \frac{e^{-c\kappa/\log\kappa} + y^{-1/\varepsilon}}{\lambda}.$$

Taking $\lambda = \kappa^{-2}$ and noticing $y \geqslant 2e^t \asymp \kappa$, we deduce

$$R \ll t/e^t. \tag{3.8}$$

Combining (3.7) and (3.8) with (3.1), we obtain

$$\Phi(t,y) \leqslant \frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^\kappa}\left\{1+O\left(\frac{t}{e^t}\right)\right\} \leqslant \Phi\big(te^{-\lambda}, y\big) \tag{3.9}$$

uniformly for $t \geqslant 1$, $y \geqslant 2e^t$ and $0 < \lambda \leqslant e^{-t}$.

On the other hand, (3.2) and (3.6) imply

$$\begin{aligned}\Phi\big(te^{-\lambda}, y\big) - \Phi(t,y) &\ll_\varepsilon \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^\kappa}\left(e^t\lambda + \frac{1}{y^{2/\varepsilon}\lambda}\right)\\ &\ll_\varepsilon \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^\kappa}e^t\lambda,\end{aligned}$$

when $\lambda \geqslant y^{-1/\varepsilon}$. Since $\Phi(te^{-\lambda}, y) - \Phi(t,y)$ is a non-decreasing function of λ, we deduce

$$\Phi\big(te^{-\lambda}, y\big) - \Phi(t,y) \ll_\varepsilon \frac{E(\kappa,y)}{\kappa\sqrt{\sigma_2}(e^\gamma t)^\kappa}\big(e^t\lambda + y^{-1/\varepsilon}\big) \tag{3.10}$$

uniformly for $t \geqslant 1$, $y \geqslant 2e^t$ and $0 < \lambda \leqslant e^{-t}$. Obviously the estimates (3.9) and (3.10) imply (1.17) and (1.18). This completes the proof of Theorem 1.

4. Proof of Theorem 2

We first establish a preliminary lemma.

Lemma 4.1. *For any integer $N \geqslant 1$, we have*

$$\phi(\sigma, y) = \sigma \log_2 \sigma + \gamma\sigma + \sigma\left\{\sum_{n=1}^{N} \frac{d_n}{(\log\sigma)^n} + O_N\big(R_N(\sigma, y)\big)\right\}$$

uniformly for $y \geqslant \sigma \geqslant 3$, where $R_N(\sigma, y)$ is defined as in (1.21) *and*

$$d_n := \int_0^\infty \frac{f(u)}{u^2}(\log u)^{n-1}\,\mathrm{d}u.$$

Proof. For $p \geqslant \sigma$, we have

$$\begin{aligned} E_p(\sigma) &= \frac{p}{p+1}\left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\}\cosh\left(\frac{\sigma}{p}\right) + \frac{1}{p+1} \\ &= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\}\cosh\left(\frac{\sigma}{p}\right) + O\left(\frac{\sigma^2}{p^3}\right) \\ &= \left\{1 + O\left(\frac{\sigma}{p^2}\right)\right\}\cosh\left(\frac{\sigma}{p}\right), \end{aligned}$$

since $\cosh(\sigma/p) \geqslant 1$ for $p \geqslant \sigma$. Thus

$$\sum_{\sigma < p \leqslant y} \log E_p(\sigma) = \sum_{\sigma < p \leqslant y} f(\sigma/p) + O(1/\log\sigma). \tag{4.1}$$

In order to treat the sum over $p \leqslant \sigma$, we write

$$E_p(\sigma) = (1 - 1/p)^{-\sigma} E_p^*(\sigma),$$

where

$$E_p^*(\sigma) := \frac{p}{2(p+1)}\left\{1 + \left(\frac{1 + 1/p}{1 - 1/p}\right)^{-\sigma} + \frac{2}{p}\left(1 - \frac{1}{p}\right)^{\sigma}\right\} \asymp 1.$$

Thus

$$\sum_{p \leqslant \sigma^{1/2}} \left|\log E_p^*(\sigma)\right| \ll \sigma^{1/2}/\log\sigma. \tag{4.2}$$

For $\sigma^{1/2} < p \leqslant \sigma$, we have $(1 - 1/p)^\sigma = e^{-\sigma/p}\{1 + O(\sigma/p^2)\}$ and

$$\begin{aligned}E_p(\sigma) &= \frac{p}{p+1}\left\{1+O\left(\frac{\sigma}{p^2}\right)\right\}\cosh\left(\frac{\sigma}{p}\right)+\frac{1}{p+1}\\ &= \left\{1+O\left(\frac{\sigma}{p^2}+\frac{1}{p}\right)\right\}\cosh\left(\frac{\sigma}{p}\right)\\ &= \left\{1+O\left(\frac{\sigma}{p^2}\right)\right\}\cosh\left(\frac{\sigma}{p}\right).\end{aligned}$$

Thus

$$\begin{aligned}E_p^*(\sigma) &= E_p(\sigma)(1-1/p)^\sigma\\ &= \left\{1+O\left(\frac{\sigma}{p^2}\right)\right\}\cosh\left(\frac{\sigma}{p}\right)e^{-\sigma/p}.\end{aligned}$$

From this and (4.2), we deduce that

$$\sum_{p\leqslant\sigma}\log E_p^*(\sigma) = \sum_{\sigma^{1/2}<p\leqslant\sigma} f(\sigma/p)+O\big(\sigma^{1/2}/\log\sigma\big),$$

which and (2.7) imply

$$\begin{aligned}\sum_{p\leqslant\sigma}\log E_p(\sigma) &= \sigma\sum_{p\leqslant\sigma}\log(1-1/p)^{-1}+\sum_{p\leqslant\sigma}\log E_p^*(\sigma)\\ &= \sigma\log_2\sigma+\gamma\sigma+\sum_{\sigma^{1/2}<p\leqslant\sigma} f\left(\frac{\sigma}{p}\right)+O\big(\sigma e^{-\sqrt{\log\sigma}}\big).\end{aligned}\tag{4.3}$$

Combining (4.1) and (4.3), we obtain

$$\phi(\sigma,y) = \sigma\log_2\sigma+\gamma\sigma+\sum_{\sigma^{1/2}<p\leqslant y} f\left(\frac{\sigma}{p}\right)+O\big(\sigma e^{-\sqrt{\log\sigma}}\big).\tag{4.4}$$

By using the prime number theorem of form

$$\pi(t) := \sum_{p\leqslant t}1 = \int_2^t\frac{\mathrm{d}v}{\log v}+O\big(te^{-2\sqrt{\log t}}\big),$$

we can prove, similar to (2.9), that

$$\sum_{\sigma^{1/2}<p\leqslant y} f\left(\frac{\sigma}{p}\right) = \sigma\left\{\sum_{n=1}^N\frac{d_n}{(\log\sigma)^n}+O_N\big(R_N(\sigma,y)\big)\right\}.\tag{4.5}$$

Now the required result follows from (4.4) and (4.5). □

Now we are ready to prove Theorem 2.

By using Lemma 4.1 and (2.17) of Lemma 2.3, we can write

$$\begin{aligned}\frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^\kappa} &= \exp\bigl\{\phi(\kappa,y)-\kappa(\gamma+\log t)+O(\log\kappa)\bigr\} \\ &= \exp\left\{\kappa\log_2\kappa+\kappa\left[\sum_{n=1}^N\frac{d_n}{(\log\kappa)^n}+O_N\bigl(R_N(\kappa,y)\bigr)\right]-\kappa\log t\right\}. \qquad (4.6)\end{aligned}$$

On the other hand, Lemma 2.1 and (1.16) imply that

$$\log_2\kappa+\sum_{n=1}^N\frac{b_n}{(\log\kappa)^n}+O_N\bigl(R_N(\kappa,y)\bigr)=\log t.$$

Thus (4.6) can be simplified as

$$\frac{E(\kappa,y)}{\kappa\sqrt{2\pi\sigma_2}(e^\gamma t)^\kappa}=\exp\left\{-\kappa\left[\sum_{n=1}^N\frac{b_n-d_n}{(\log\kappa)^n}+O_N\bigl(R_N(\kappa,y)\bigr)\right]\right\}.$$

A simple integration by parts shows $b_n-d_n=a_n$. This completes the proof.

5. Proof of Corollary 3

By using (2.15), we have

$$\begin{aligned}\sum_{n=1}^N\frac{a_n}{(\log\kappa)^n} &= \sum_{n=1}^N\frac{a_n}{t^n}\left\{1-\sum_{i=1}^{N-n}\frac{\gamma_{i-1}}{t^i}+O_N\left(\frac{R^*_{N-n-1}(t,y)}{t}\right)\right\}^{-n} \\ &= \sum_{n=1}^N\frac{\rho_n}{t^n}+O_N\left(\frac{R^*_{N-2}(t,y)}{t^2}\right), \qquad (5.1)\end{aligned}$$

where the ρ_n are constants. In particular we have $\rho_1=a_1=1$ and $\rho_2=\gamma_0+a_2$.

Now Theorem 2, (2.10) and (5.1) imply the result of corollary with

$$a_1^*=\rho_1=1,\qquad a_n^*=\rho_n+\sum_{i=1}^{n-1}c_i\rho_{n-i}\quad(n\geqslant 2).$$

This completes the proof of Corollary 3.

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